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New solutions for the short-time analysis of geothermal vertical boreholes

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Abstract

Many models, either numerical or analytical, have been proposed to analyse the thermal response of vertical heat exchangers that are used in ground coupled heat pump systems (GCHP). In both approaches, most of the models are valid after few hours of operation since they neglect the heat capacity of the borehole. This is valid for design purposes, where the time of interest is in the order of months and years. Recently, the short time response of vertical boreholes became a subject of interest. In this paper, we present a new analytical approach to treat this problem. It solves the exact solution for concentric cylinders and is a good approximation for the familiar U-tube configuration.

Keywords: Ground source heat pump; Ground heat exchanger; Analytical solution; Short time response

1. Introduction

Ground coupled heat pump systems exist for many years and the concept is widely accepted as one of the best renewable energy technology. Until recently, the initial cost of these systems hindered their growth, especially for residential purposes. The recent increase in energy costs and the need to reduce greenhouse effect gases will hopefully increase their use. An important part of the cost of the system is the heat exchanger in the ground. The efficiency of such a system depends on how much heat is extracted (in winter) or rejected (in summer) in the ground. In the case of vertical heat exchangers, in most configurations, the fluid passes through U-tubes in the form shown in Fig. 1. The borehole is then filled by a material called *grout*. Typical boreholes have one or two U-tubes and may have more than one borehole. For the design of vertical boreholes, we are interested in the heat transfer between the working fluid and the ground. There have been a number of models based on some analytical solutions like proposed by Ingersoll [1] or Kavanaugh [2] or numerical solutions like the one proposed by Eskilson's [3] and Hellstrom [4] that were used for designing vertical boreholes used in GCHP systems. As we will see in the following section, these models often neglect or use a oversimplified approach to treat the borehole short time behaviour. In many design programs, this is not important since the time of interest is in the order of months and years when this effect is negligible. For example, Eskilson's proposed the following limit for its model:

$$t > \frac{5r_b^2}{\alpha} \qquad (1)$$

For a typical borehole, this value can be in the order of 3–6 h. In recent years, short time simulations of GCHP systems in the order of minutes ask for a more precise model for these time intervals. Yavuzturk and Spitler [5] were one of the first to analyse the problem. They actually solved the numerical heat diffusion problem in the ground taking into account the heat capacity of the pipe and the grout. Sutton et al. [6] and Young [7] both proposed recently some analytical solutions for this problem that will be described in the following section.

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Nomenclature

C_p	specific heat (J kg^{-1} K^{-1})
E_1	exponential integral
G	G-function
H	borehole length (m)
$\widetilde{H}$	non-dimensional convection coefficient (Biot number)
k	thermal conductivity (W m^{-1} K^{-1})
$\dot{m}$	mass flow rate (kg s^{-1})
q'	heat flux per unit length (W m^{-1})
r	radial coordinate (m)
$\tilde{r}$	non-dimensional radius $r/r_{\rm b}$
R'	unit length thermal resistance (K m W^{-1})
$\tilde{t}$	non-dimensional time $\alpha t/r^2$ (Fourier number)
T	temperature (K)
z	axial coordinate (m)

Greek symbols

α	thermal diffusivity (m^2 s^{-1})
δ	$r_{\rm b}/r_{\rm e}$
γ	$\sqrt{\alpha_g/\alpha_s}$

Subscripts

b	at the borehole radius
e	outer radius of the inner tube for concentric cylinders
f	associated to the calorimetric fluid
fi	fluid entrance
fo	fluid outlet
g	associated to the grout material
p	at the pipe radius
s	associated to the soil material

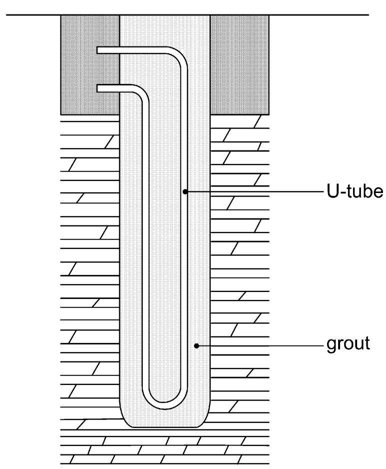

Fig. 1. Ground heat exchanger.

In this paper, we propose a new analytical approach to the short-time response of a vertical borehole. It is based on the exact solution of the heat equation in a compound infinite medium. It will be seen that it is a very good approximation for the U-tube shape configuration found in most ground heat exchangers. It shares the flexibility associated with analytical solutions and the results will be compared with the other proposed solutions.

2. Existing models

The classical approach for the analysis of vertical heat exchangers is to model the inside of the borehole as a simple thermal resistance. The mean fluid temperature, $T_{\rm f}(t)$, can be computed using the following simple expression:

$$T_{\rm f}(t) - T_{\rm b}(t) = q'_{\rm b}(t)R'_{\rm b} \tag{2}$$

where $R'_{\rm b}$ is the unit length resistance of the borehole which considers the convection between the fluid and its wall, the conduction in the tube wall and the conduction in the grout. Since the heat is transferred to a moving fluid, this temperature is in fact a mean fluid temperature:

$$T_{\rm f}(t) = \frac{T_{\rm fi}(t) + T_{\rm fo}(t)}{2} \tag{3}$$

and the inlet and outlet temperatures can be computed from the following relations:

$$\frac{q'_{\rm b}(t)L}{\dot{m}C_p} = T_{\rm fi}(t) - T_{\rm fo}(t) \tag{4}$$

$$T_{\rm fo}(t) = T_{\rm f}(t) + \frac{q'_{\rm b}(t)H}{2\dot{m}C_p} \tag{5}$$

The borehole temperature $T_{\rm b}(t)$ is found from the transient thermal response in the infinite ground. Analytical or numerical solutions are used to find this temperature. The mostly used numerical solution is the DST (duct storage model) proposed by Hellström [4], which is implemented in the TRNSYS simulation software package. One of the two major analytical solutions is known as the line source model [1]:

$$T(\tilde{r},\tilde{t}) - T_0 = \frac{q'_{\rm b}}{4\pi k}\int_{\tilde{r}^2/4\tilde{t}}^{\infty}\frac{{\rm e}^{-u}}{u}{\rm d}u = \frac{q'_{\rm b}}{4\pi k}E_1(\tilde{r}/(4\tilde{t})) \tag{6}$$

where E_1 is the exponential integral, $\tilde{r} = r/r_{\rm b}, \tilde{t} = \alpha t/r_{\rm b}^2 =$ Fouriernumber and T_0 is the undisturbed ground temperature. The other well-known solution is the cylindrical heat source (CHS) solution given by Carslaw and Jaeger [8]:

$$T(\tilde{r},\tilde{t}) - T_0 = \frac{q'_{\rm b}}{k}\underbrace{\frac{1}{\pi^2}\int_0^{\infty}\frac{e^{-\beta^2\tilde{t}}-1}{\beta^2(J_1^2(\beta)+Y_1^2(\beta))}[J_0(\tilde{r}\beta)Y_1(\beta) - J_1(\beta)Y_0(\tilde{r}\beta)]{\rm d}\beta}_{G(\tilde{r},\tilde{t})} \tag{7}$$

This solution is often referred in the literature as the *G-function*. This is basically the exact solution of the following mathematical problem:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r} \tag{8}$$

for the domain $r>r_b$, $t>0$, and the following boundary conditions

$$T(r,0)=T_0,\quad -k\frac{\partial T}{\partial r}\bigg|_{r=r_{\rm b}}=q''_{\rm b}(t)=\frac{q'_{\rm b}(t)}{2\pi r_{\rm b}} \tag{9}$$

In both cases, the borehole temperature is found by evaluating the given expression at $\tilde{r}=1$. In this approach, the evaluation of the borehole resistance $R'_{\rm b}$ is crucial. Young [7] gives a very good survey of these methods. The two major approaches used are the one proposed by Paul [9] and by Hellström [4]. The first one is given in terms of shape factors.

$$R'_{\rm b}=\frac{1}{Sk_{\rm grout}} \tag{10}$$

where the shape factor S is given by

$$S=\beta_0\left(\frac{r_{\rm b}}{r_0}\right)^{\beta_1} \tag{11}$$

The values of the two parameters β_0, β_1 are found from correlations from numerical results. They are function of the so-called shank spacing. The values are given in Table 1.

Hellstrom [4] proposed two methods, one based on the line source for composite region, which is an approximation of the more general multipole method proposed by Bennet et al. [10]. Since the results from the line source are very similar to the multipole expansion, this approach is usually used in the DST model, the main reference for vertical ground heat exchangers. The line source method offers two main advantages: it takes into account the different conductivities of the grout and the soil, and it is applicable for multiple U-tubes in the boreholes. Whatever method is used to model the thermal resistance of the borehole, the fact to replace the borehole by a simple thermal resistance gives wrong results for the initial response of the whole system. Indeed, two problems will arise from the application of (2) and (3). First, as soon as a heat flow is imposed between the fluid and the ground, a ΔT given by (2) will be felt whereas from physical reasoning, we would expect a certain time for this to build-up. Secondly, the thermal resistance found from the steady state analysis of the grout will be overestimated from the one felt in reality. For these reasons, the short time response of vertical boreholes has been a subject of interest recently.

Table 1
Paul curve fit parameters for (10) and (11)

	A	B	C
β_0	14.450872	17.44268	21.90587
β_1	−0.8176	−0.605154	−0.3796

As we said in Section 1, Yavuzturk and Spitler [5] solved the numerical heat diffusion in the ground taking into account the heat capacity of the pipe and the grout. Their numerical results were then expressed in term of a *short-time g-function*, a concept introduced by Eskilson [3] and defined as

$$T_{\rm b}-T_0=\frac{q'_{\rm b}}{2\pi k}g(t/ts,r_{\rm b}/H) \tag{12}$$

where H is the height of the borehole and $t_{\rm s}=H^2/(9\alpha)$, the reference time for their analysis. This time is actually a typical time when axial conduction effects become important and it is generally very long. The problem with numerical results is that we need the program if we want to know the effect of various material properties like grout conductivity and heat capacity. Sutton et al. [6] proposed the following algorithm to take into account the short time behaviour on the borehole. They express their results in terms of the pipe temperature $T_{\rm p}(t)$, which is the outside temperature of the U-tube pipe. Although they refer to this temperature as the fluid temperature, this is strictly true if we neglect the convection thermal resistance and the pipe conduction resistance. They first use the classical cylindrical source solution but with the material properties of the grout and they replace the U-tube arrangement by an equivalent cylinder, $r_{\rm eq}$ for the radius of the cylinder source. This solution can be modeled by the following relation:

$$T_{\rm p}(t)-T_0=\frac{q'_{\rm b}}{k_{\rm g}}G(1,\alpha_{\rm g}t/r_{\rm eq}^2)=q'_{\rm b}R'_{\rm st}(t) \tag{13}$$

A transition time $\tau_{\rm grout}$ is defined as

$$R'_{\rm st}(\tau_{\rm grout})=R'_{\rm b} \tag{14}$$

and the final solution is given by

$$T_{\rm p}(t)-T_0=\begin{cases}\frac{q'_{\rm b}}{k_{\rm g}}G(1,\alpha_{\rm g}t/r_{\rm eq}^2), & t<\tau_{\rm grout}\\ q'_{\rm b}\left(\frac{G(1,\alpha_{\rm s}t/r_{\rm b}^2)}{k_{\rm s}}+R'_{\rm b}\right), & t>\tau_{\rm grout}\end{cases} \tag{15}$$

In their approach they defined an equivalent radius, a technique which is very often used in the analytical treatment of U-tubes ground exchangers. Bose [11] was one of the first to propose the following value:

$$r_{\rm eq}=\sqrt{2}r_0 \tag{16}$$

Kavanaugh [2] suggested to include a correction factor to this value:

$$r_{\rm eq}=\sqrt{2}r_0+x \tag{17}$$

where x is the shank spacing between the tubes (Fig. 2). The value proposed by Sutton et al. is different. They propose the following value for the equivalent radius:

$$\frac{\log\left(\frac{r_{\rm b}}{r_{\rm eq}}\right)}{2\pi k_{\rm g}}=R'_{\rm b} \tag{18}$$

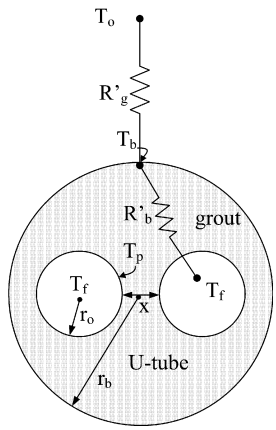

Fig. 2. Thermal resistances.

In their work they used the solution proposed by Paul for R'_b. We will come back to the solution proposed by Sutton et al. in Section 3 (see Fig. 3).Young [7] propose a different approach to evaluate the short time response of the borehole. It uses the exact solution of the following problem: the heat from or to the ground is exchanged with two perfect conductors separated by a contact resistance. The physical problem is illustrated in Fig. 4 and the solution is also given by Carslaw and Jaeger [8, p. 344]. It is referred as the *buried cable* solution since it was solved in the context of a buried electrical cable. In this context, the two perfect conductors are the current-carrying core (in copper or aluminum) separated from a metal sheath and the contact thermal resistance R' represent the electrical insulation between these two conductors. The solution of the problem is given by Eq. (8) but now with the following boundary conditions:

$$T(r,0) = T_0,$$
$$2\pi r_b k_s \frac{\partial T}{\partial r}\bigg|_{r=r_b} = \frac{T_2 - T_1}{R'} + \underbrace{A_{c2}\rho_2 C_{p_2}}_{S_2} \frac{\partial T_2}{\partial t} \tag{19}$$

Another heat balance is needed for the core

$$q'_b = \frac{T_1 - T_2}{R'} + \underbrace{A_{c1}\rho_1 C_{p_1}}_{S_1} \frac{\partial T_1}{\partial t} \tag{20}$$

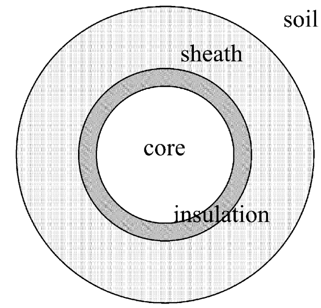

Fig. 4. Buried cable model.

where A_c stands for the cross-section area or the volume per unit length of the cylinder. The problem is solved by Laplace transform and the solution is given by

$$T_p(\tilde{t}) - T_0 = \frac{q'_b}{k_s} \underbrace{\frac{2\alpha_1^2\alpha_2^2}{\pi^3}\int_0^\infty \frac{(1 - e^{-\beta^2\tilde{t}})d\beta}{\beta^3\Delta(\beta)}}_{G_{bc}(\tilde{t})} \tag{21}$$

where

$$\alpha_1 = \frac{2\pi r_b^2 \rho_s C_{p_s}}{S_1}, \quad \alpha_2 = \frac{2\pi r_b^2 \rho_s C_{p_s}}{S_2}, \quad \widetilde{U} = 2\pi R' k_s \tag{22}$$

$$\Delta(\beta) = (\beta(\alpha_1 + \alpha_2 - \tilde{U}\beta^2)J_0(\beta) - \alpha_2(\alpha_1 - \tilde{U}\beta^2)J_1(\beta))^2 + (\beta(\alpha_1 + \alpha_2 - \tilde{U}\beta^2)Y_0(\beta) - \alpha_2(\alpha_1 - \tilde{U}\beta^2)Y_1(\beta))^2 \tag{23}$$

Even though the solution was given in the context of a buried cable, Young successfully applied this solution to the vertical heat exchanger where the core was replaced by an annular of fluid with heat applied at its inner radius, the metal sheath represents the thermal capacity of the grout and the contact resistance models the steady-state thermal resistance. In his work he used the multipole expression of Bennet et al. for the value of the resistance. To have better results, he modified the model by moving part of the grout thermal capacity from the outside of the thermal resistance to the inside of the thermal resistance (in the core region) through a grout allocation factor (GAF) defined as

$$S_{1f} = S_1 + S_2 f, \quad S_{2f} = S_2(1 - f) \tag{24}$$

with $0 < f < 1$ being the GAF. We will discuss the results from this model in Section 3. Finally, some numerical

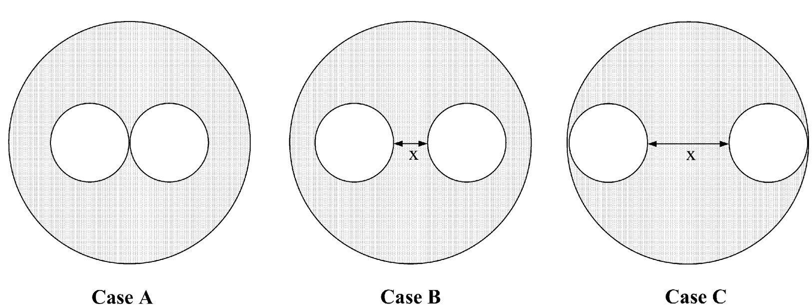

Fig. 3. Paul shank spacing.

models now attack the short time response of a buried heat exchanger. This is the case, for example of the TRNSYS TYPE451 component [12]. In this paper we present, what we think is a new contribution to the analytical description of the short time response of a vertical heat exchanger. Although numerical solutions are easily obtained with the use of modern computers, a good analytical description of the physical problem is always a good tool to add flexibility for different configurations in the design step of the borehole field.

3. New solution of the short time response factor

Since both analytical solutions given in the previous section are approximations of the following problem:

$$\frac{1}{\alpha_g}\frac{\partial T_1}{\partial t}=\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\frac{\partial T_1}{\partial r} \tag{25}$$

for the domain $r_e < r < r_b$, $t > 0$, where T_1 represent the grout temperature.

$$\frac{1}{\alpha_s}\frac{\partial T_2}{\partial t}=\frac{\partial^2 T_2}{\partial r^2}+\frac{1}{r}\frac{\partial T_2}{\partial r} \tag{26}$$

for the domain $r_b < r$, $t > 0$, where T_2 is the soil temperature. A natural choice is to solve the real problem exactly. As all the other analytical models, it will be an approximation for the general U-tube configuration. Since this problem does not seem to be presented in the classical books on heat conduction like the one from Carslaw and Jaeger [8] or the one by Osizick [13], it is believed to be a new contribution to the field. It will be seen that, although the expressions become rather complicated, they can be written in closed form and computed easily with the availability of mathematical softwares. We will treat two different cases: the first one is the classical approach where the total heat per unit length q_b' is given. The second case is when the mean fluid temperature is known and heat is transferred through convection to the pipe. In the first case, comparison to the two other analytical solutions is easily done.

3.1. Heat flux imposed

In this case, besides the governing Eqs. (25) and (26), the following boundary conditions are imposed

$$T_1(r,0)=T_2(r,0)=T_0,\quad -k_g\left.\frac{\partial T_1}{\partial r}\right|_{r=r_e}=q_b''(t) \tag{27}$$

$$T_1(r_b,t)=T_2(r_b,t),\quad -k_g\left.\frac{\partial T_1}{\partial r}\right|_{r=r_b}=-k_s\left.\frac{\partial T_2}{\partial r}\right|_{r=r_b} \tag{28}$$

Although lengthy, the procedure is classical as it is explained in Carslaw and Jaegers [8]. The solution is found using the Laplace transform: (25) and (26) become

$$\frac{d^2\overline{T}}{dr^2}+\frac{1}{r}\frac{d\overline{T}}{dr}-q^2\overline{T}=0 \tag{29}$$

where $\overline{T}$ is the Laplace transform of $T-T_0$ and $q^2=s/\alpha$. The solutions of (29) for both regions are

$$\overline{T}_1=c_1K_0(q_1r)+c_2I_0(q_1r) \tag{30}$$

$$\overline{T}_2=c_3K_0(q_2r)+c_4I_0(q_2r) \tag{31}$$

As usual, $c_4=0$ to keep the temperature finite. The other constants are found from the other boundary conditions. Solving for a step change of the heat flux, $q_b'(t)=q_b'u(t)$ we find

$$-k_g\frac{d\overline{T}_1(q_1r_e)}{dr}=(c_1K_1(q_1r_e)-c_2I_1(q_1r_e))q_1k_g=\frac{q_b''}{s} \tag{32}$$

$$c_1K_0(q_1r_b)+c_2I_0(q_1r_b)=c_3K_0(q_2r_b) \tag{33}$$

$$(c_1K_1(q_1r_b)-c_2I_1(q_1r_b))q_1k_g=c_3K_1(q_2r_b)q_2k_s \tag{34}$$

The solution for $\overline{T}_1$ becomes

$$\overline{T}_1(s,r)=\frac{q_b''}{q_1k_gs\,\mathrm{den}}\big[(K_0(q_2r_b)I_1(q_1r_b)+\tilde{k}\gamma K_1(q_2r_b)I_0(q_1r_b))K_0(q_1r) +(K_0(q_2r_b)K_1(q_1r_b)-\tilde{k}\gamma K_1(q_2r_b)K_0(q_1r_b))I_0(q_1r)\big] \tag{35}$$

with

$$\mathrm{den}=(K_0(q_2r_b)I_1(q_1r_b)+\tilde{k}\gamma K_1(q_2r_b)I_0(q_1r_b))K_1(q_1r_e) -(K_0(q_2r_b)K_1(q_1r_b)-\tilde{k}\gamma K_1(q_2r_b)K_0(q_1r_b))I_1(q_1r_e)$$

$$\text{and } \gamma=\sqrt{\alpha_g/\alpha_s},\ \tilde{k}=k_s/k_g \tag{36}$$

As usual the inverse found from the inversion theorem on the contour is shown in Fig. 6.

$$T_1(t,r)=T_0+\frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty}e^{st}\overline{T}_1(s,r)\,ds \tag{37}$$

It can be shown that there are no poles within or on the contour (Appendix A). So the solution is given by

$$T_1(t,r)=T_0+\frac{1}{2\pi j}\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5}e^{st}\overline{T}_1(s,r)\,ds \tag{38}$$

As it is, the solution would be singular on Γ_3, so we find the solution for

$$s\overline{T}_1=\mathscr{L}\left(\frac{\partial T_1}{\partial t}-(T_1(0,r)-T_0)\right)=\mathscr{L}\left(\frac{\partial T_1}{\partial t}\right)$$

$$\frac{\partial T_1}{\partial t}=\frac{1}{2\pi j}\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5}e^{st}s\overline{T}_1(s,r)\,ds \tag{39}$$

It is easy to verify that the contributions on Γ_1, Γ_3, Γ_5 are zero. The remaining will give the following solution:

$$T_1(\tilde{r},\tilde{t})-T_0=\frac{q_b'}{k_g}\underbrace{\frac{4\tilde{k}}{\pi^4\delta^2}\int_0^\infty\frac{(Y_0(\beta\tilde{r})J_1(\beta)-J_0(\beta\tilde{r})Y_1(\beta))\left(1-e^{-\beta^2\tilde{t}}\right)}{\beta^4(\phi^2+\psi^2)}d\beta}_{G_{st_q}(\tilde{r},\tilde{t})} \tag{40}$$

with

$$\phi = Y_1(\beta)[Y_0(\beta\delta\gamma)J_1(\beta\delta) - Y_1(\beta\delta\gamma)J_0(\beta\delta)\tilde{k}\gamma] - J_1(\beta)[Y_0(\beta\delta\gamma)Y_1(\beta\delta) - Y_1(\beta\delta\gamma)Y_0(\beta\delta)\tilde{k}\gamma] \quad (41)$$

$$\psi = J_1[(\beta)J_0(\beta\delta\gamma)Y_1(\beta\delta) - J_1(\beta\delta\gamma)Y_0(\beta\delta)\tilde{k}\gamma] - Y_1(\beta)[J_0(\beta\delta\gamma)J_1(\beta\delta) - J_1(\beta\delta\gamma)J_0(\beta\delta)\tilde{k}\gamma] \quad (42)$$

and

$$\tilde{r} = r/r_e, \quad \tilde{t} = \alpha_g t/r_e^2, \quad \delta = r_b/r_e$$

The notation G_{st_q} stands for the *short time G-function for fixed flux*. Before giving numerical values, we might verify that the solution is indeed the classical solution when the two mediums are the same. Putting $\tilde{k} = \gamma = 1$ in the last solution we obtain

$$\phi = \frac{2Y_1(\beta)}{\pi\beta\delta}, \quad \psi = \frac{2J_1(\beta)}{\pi\beta\delta} \quad (43)$$

and (40) gives the classical *G-function* given by (7). In most cases, we are mainly concern with the temperature at the boundary. In that case (40) becomes:

$$T_p(\tilde{t}) = T_1(1,\tilde{t}) = T_0 + \frac{q_b'}{k_g}\underbrace{\frac{8\tilde{k}}{\pi^5\delta^2}\int_0^\infty \frac{\left(1 - e^{-\beta^2\tilde{t}}\right)}{\beta^5(\phi^2 + \psi^2)}d\beta}_{G_{st_q}(1,\tilde{t})} \quad (44)$$

3.2. Convection

In this case, we solve again the Eqs. (25) and (26) but with the following boundary conditions

$$T_1(r,0) = T_2(r,0) = T_0, \quad -k_g\frac{\partial T_1}{\partial r}\bigg|_{r=r_e} = h(T_f - T_1(r_e,t)) \quad (45)$$

$$T_1(r_b,t) = T_2(r_b,t), \quad -k_g\frac{\partial T_1}{\partial r}\bigg|_{r=r_b} = -k_s\frac{\partial T_2}{\partial r}\bigg|_{r=r_b} \quad (46)$$

We will first analyse the case where T_f is constant. In this case the solution will be given in terms of $\overline{T}$, the Laplace transform of $T - T_f$. The solutions for both regions are now:

$$\overline{T}_1 = c_1K_0(q_1r) + c_2I_0(q_1r) + \frac{T_0 - T_f}{s} \quad (47)$$

$$\overline{T}_2 = c_3K_0(q_2r) + \frac{T_0 - T_f}{s} \quad (48)$$

Following the same procedure as the last section, the solution for $\overline{T}_1$ becomes

$$\overline{T}_1(s,r) = \frac{T_0 - T_f}{s}\left[1 - \frac{K_0(q_2r)}{q_1r_b(\xi_1\rho_1 - \xi_2\rho_2)}\right] \quad (49)$$

with

$$\xi_1 = K_0(q_2r_b)I_1(q_1r_b) + \tilde{k}\gamma K_1(q_2r_b)I_0(q_1r_b) \quad (50)$$

$$\xi_2 = K_0(q_2r_b)K_1(q_1r_b) - \tilde{k}\gamma K_1(q_2r_b)K_0(q_1r_b) \quad (51)$$

$$\rho_1 = \frac{q_1k_g}{h}K_1(q_1r_e) + K_0(q_1r_e) \quad (52)$$

$$\rho_2 = \frac{q_1k_g}{h}I_1(q_1r_e) - I_0(q_1r_e) \quad (53)$$

Again, there are no poles inside the contour (Appendix A). The solution is given by

$$T_1(t,r) - T_f = \frac{T_0 - T_f}{2\pi j}\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5} e^{st}\frac{K_0(q_2r)}{sq_1r_b(\xi_1\rho_1 - \xi_2\rho_2)}ds \quad (54)$$

Again the contributions on Γ_1 and Γ_5 are zero. However, the Γ_3 will give a contribution which will cancel the T_0 term. The remaining parts of the contour will give the following solution:

$$T(\tilde{r},\tilde{t}) - T_f = (T_0 - T_f)\underbrace{\frac{8\tilde{k}\widetilde{H}}{\pi^3\delta^2}\int_0^\infty \frac{\Upsilon e^{-\beta^2\tilde{t}}}{\beta^3(\phi_c^2 + \psi_c^2)}d\beta}_{G_{st_c}(\tilde{r},\tilde{t})} \quad (55)$$

with

$$\Upsilon = \beta[Y_0(\beta\tilde{r})J_1(\beta) - J_0(\beta\tilde{r})Y_1(\beta)] + \tilde{k}\gamma\widetilde{H}[Y_0(\beta\tilde{r})J_0(\beta) - J_0(\beta\tilde{r})Y_0(\beta)] \quad (56)$$

$$\phi_c = [\widetilde{H}J_0(\beta) + \beta J_1(\beta)][J_0(\beta\delta\gamma)Y_1(\beta\delta) - J_1(\beta\delta\gamma)Y_0(\beta\delta)\tilde{k}\gamma] - [\widetilde{H}Y_0(\beta) + \beta Y_1(\beta)][J_0(\beta\delta\gamma)J_1(\beta\delta) - J_1(\beta\delta\gamma)J_0(\beta\delta)\tilde{k}\gamma] \quad (57)$$

$$\psi_c = [\widetilde{H}J_0(\beta) + \beta J_1(\beta)][Y_0(\beta\delta\gamma)Y_1(\beta\delta) - Y_1(\beta\delta\gamma)Y_0(\beta\delta)\tilde{k}\gamma] - [\widetilde{H}Y_0(\beta) + \beta Y_1(\beta)][Y_0(\beta\delta\gamma)J_1(\beta\delta) - Y_1(\beta\delta\gamma)J_0(\beta\delta)\tilde{k}\gamma]$$

$$\text{and } \widetilde{H} = \frac{hr_e}{k_g} \quad (58)$$

The notation G_{st_c} stands for the *short time G-function for convection*. Again we can compare this expression when the two mediums are the same. Putting $\tilde{k} = \gamma = 1$ in the last solution we obtain

$$\Upsilon = Y_0(\beta\tilde{r})(\beta J_1(\beta) + \widetilde{H}J_0(\beta)) - J_0(\beta\tilde{r})(\beta Y_1(\beta) + \widetilde{H}Y_0(\beta)) \quad (59)$$

$$\phi_c = \frac{2(\widetilde{H}J_0(\beta) + \beta J_1(\beta))}{\pi\beta\delta}, \quad \psi_c = \frac{2(\widetilde{H}Y_0(\beta) + \beta Y_1(\beta))}{\pi\beta\delta} \quad (60)$$

and (55) gives the classical solution given by Carslaw and Jaeger's [8, p. 337]. As usual, we are mostly interested in the solution at the boundary which will be given by the following expression:

$$T_p(1,\tilde{t}) = T_1(1,\tilde{t}) = T_f + (T_0 - T_f)\underbrace{\frac{16\tilde{k}\widetilde{H}}{\pi^4\delta^2}\int_0^\infty \frac{e^{-\beta^2\tilde{t}}}{\beta^3(\phi_c^2 + \psi_c^2)}d\beta}_{G_{st_c}(1,\tilde{t})} \quad (61)$$

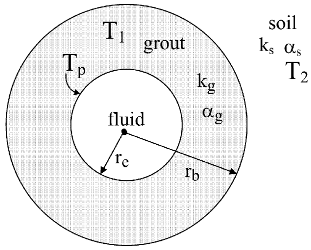

Fig. 5. Concentric cylinder.

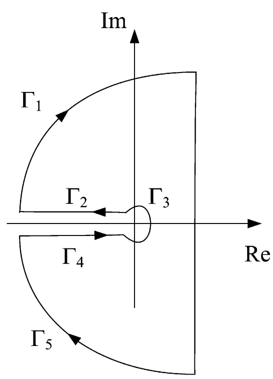

Fig. 6. Contour used for (36).

4. Numerical results

The analytical solutions presented in the previous section were compared with numerical solutions. All the numerical calculations were done with the Comsol® finite element software. Mesh refinement was done until no variation in the results was observed. Two series of numerical tests were carried out. In the first case, the solution was given for two concentric cylinders, one finite, representing the grout, and a infinite cylinder representing the soil (Fig. 5). In the finite element program, the infinite cylinder was replaced by a finite one where undisturbed temperature was imposed. The outer radius was augmented until no variation in the results was observed. In the second case, the solution was compared with a real single U-tube configuration. In the first case, our model is the exact solution of the problem and no variation with the numerical results is expected. It is basically a numerical validation of the expressions found. In the second case, the model is an approximation of the real case and the new model is compared with some other analytical approximate expressions.

4.1. Concentric cylinders heat flux imposed

Although (44) and (61) look rather repulsive, the availability of powerful interface like matlab® allows their implementation in a straightforward way. The solutions (44) and (61) are then the real analytical solutions of the problem and both results should be the same. The two different cases, fixed flux and imposed convection are compared. Results for different conductivities and heat capacity will be shown. On the figures we also compare the new solution with the "buried cable" solution introduced by Carslaw and Jaeger and proposed by Young [7] and also the compound solution of Sutton et al. [6]. Before giving the results, some comments should be given. First of all, in the model proposed by Young, the capacity of the fluid was taken into account. Actually the total fluid capacity could include fluid from outside the borehole in a factor that he called a *fluid multiplication factor* that takes into account the fluid outside the borehole, like the one in reservoirs. The main argument with that approach is to simulate the time needed by the heat source to bring the thermal fluid at a given temperature. Doing so he considers the borehole, the earth and the fluid as a closed system. In our mind, we prefer analysing the fluid as an open system with a given temperature. The initial time needed to heat that fluid would eventually be modeled outside the borehole model. The modeling of the whole system, earth heat exchanger, heat pump reservoirs, etc. will be presented in another paper. In our results, we only consider the heat capacity of the grout. However, we keep the two conductors model in order to have better results. This can be seen as choosing the following values for S_{1f} and S_{2f}:

$$S_{1f} = S_2 f, \quad S_{2f} = S_2(1-f) \tag{62}$$

Since we used this model only for comparison purposes, we did not study the effect of the grout allocation factor (f), so we kept $f = 0.5$ in all of our computations. The author also proposed a logarithmic extrapolation procedure to improve the accuracy of the model. This aspect was not considered here.

Secondly, when we tried the model proposed by Sutton et al., we observed a step in the temperature response when the solution changed from the two modes: short time, long time. This problem seems to be produced by the choice of the transition time (14). A better choice was given by the following relation:

$$R'_{st}(\tau_{grout}) = \frac{G(1, \alpha_s \tau_{grout}/r_b^2)}{k_s} + R'_b \tag{63}$$

This choice is also more coherent with (15). In the first results, a constant heat flux was imposed on the inner boundary. The solution is given by (44). As it is seen on Fig. 7a, (44) is the exact solution of the problem whereas the two other solutions simulate well the problem for large time. The buried cable model is indeed better than the classical approach but deviate from the solution at very short time as it is observed by Young. In his work Young [7] proposes to improve the accuracy by adjusting the Grout Allocation Factor and by using a logarithmic extrapolation from the numerical solution. As we said, these aspects were not investigated here. We observe that the solution proposed by Sutton et al. gives very good results for small value of time. This is normal, since the inner core does not see the ground at the beginning. We can observe some deviation when the two regimes change. However, in the second test, the grout conductivity value is larger than the soil. In that

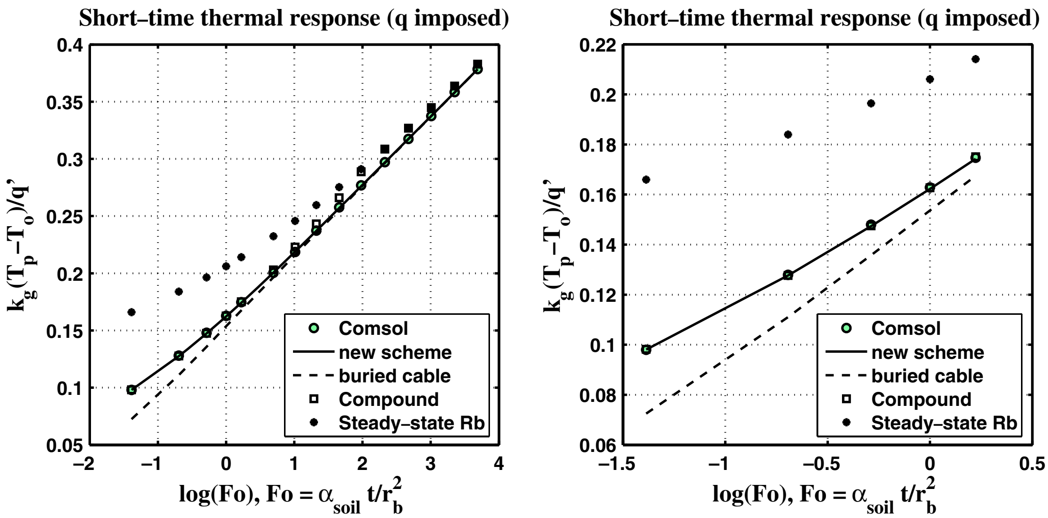

Fig. 7a. Temperature response (concentric cylinder, fixed flux) $\tilde{k} = 1.33$, $\gamma^2 = 1.5$.

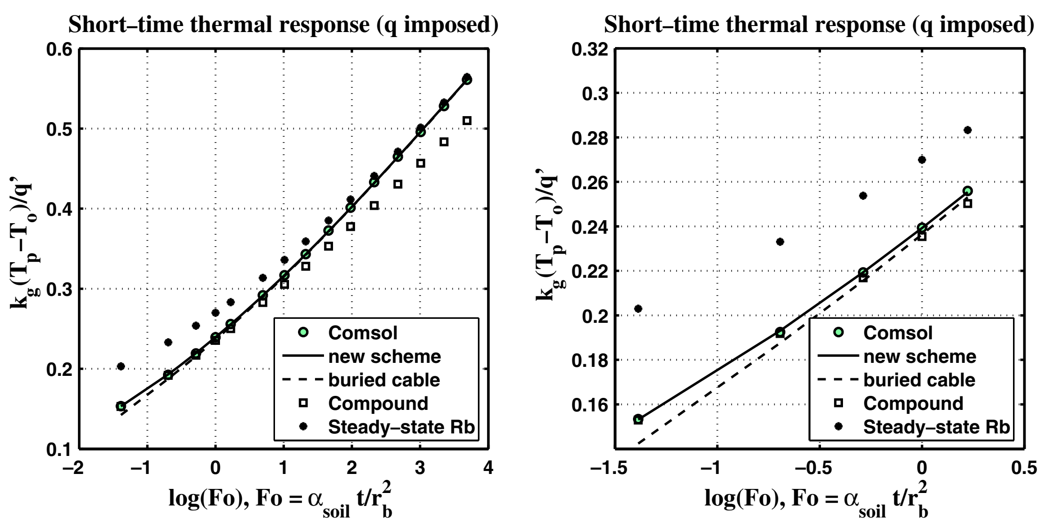

Fig. 7b. Temperature response (concentric cylinder, fixed flux) $\tilde{k} = 0.8$, $\gamma^2 = 0.75$.

case, we observe in Fig. 7b that the *compound solution* deviates as time increases. In fact, in that case, τ_{grout} given by (63) is infinite. For this reason we did not continue with more investigations with this approach.

4.1.1. Arbitrary heat function

Before analysing the results for mixed boundary conditions, we will need to look at the temperature variation when the heat flux varies in time. The classical way to treat the problem is to express the arbitrary heat function into a series of step functions [5,14]. For example, the temperature using our short-time response factor, would be given by the following expression:

$$T_{\rm p}(\tilde{t}) = T_0 + \sum_{i=1}^{N} \frac{q'_{\mathrm{b}i} - q'_{\mathrm{b}i-1}}{k_{\mathrm{g}}} G_{\mathrm{st}_q}(1, \tilde{t} - \tilde{t}_{i-1}) \tag{64}$$

The same procedure can easily be used with (21) instead. Application of (15) can also be extended but in that case, it is a little bit more laborious. It is known that application of (64) can be very time consuming. Several methods have been proposed to increase the speed of calculation [5,14,15]. We will come back to this aspect in a latter section but for now we will keep (64) as it is to treat the temperature response when the fluid temperature is imposed.

4.2. Concentric cylinders when convection with a given fluid temperature

In the second series of tests, a fluid temperature with a given convection coefficient was imposed. The exact solution is then given by (61). The other models however cannot be used directly. In that case, since the fluid temperature is kept constant, the heat flux will vary with time and (64) must be used. It must however be modified since the variable heat flux is not known. Any response factor can be used ,we will describe the procedure with (21). The following algorithm will be used:

4.2.1. Initialization

At $t = 0$, we assume $T_p(0) = T_0$

At $t = 1 \cdot \Delta\tilde{t}$, using (64)

$$T_p(\Delta\tilde{t}) - T_0 = q'_b(1)G_{bc}(\Delta\tilde{t})/k_s = hP(T_f - T_p(\Delta\tilde{t}))G_{bc}(\Delta\tilde{t})/k_s \quad (65)$$

$$\Rightarrow T_p(\Delta\tilde{t}) - T_0 = \frac{(T_f - T_0)hPG_{bc}(\Delta\tilde{t})/k_s}{1 + hPG_{bc}(\Delta\tilde{t})/k_s} \quad (66)$$

$$\Rightarrow q'_b(1) = hP(T_f - T_p(\Delta\tilde{t})) \quad (67)$$

At $t = 2 \cdot \Delta\tilde{t}$, using (64)

$$T_p(2\Delta\tilde{t}) - T_0 = q'_b(2)G_{bc}(\Delta\tilde{t})/k_s + q'_b(1)(G_{bc}(2\Delta\tilde{t}) - G_{bc}(\Delta\tilde{t}))/k_s = q'_b(1)(G_{bc}(2\Delta\tilde{t}) - G_{bc}(\Delta\tilde{t}))/k_s + hP(T_f - T_p(2\Delta\tilde{t}))G_{bc}(\Delta\tilde{t})/k_s \quad (68)$$

$$\Rightarrow T_p(2\Delta\tilde{t}) - T_0 = \frac{q'_b(1)(G_{bc}(2\Delta\tilde{t}) - G_{bc}(\Delta\tilde{t}))/k_s + (T_f - T_0)hPG_{bc}(\Delta\tilde{t})/k_s}{1 + hPG_{bc}(\Delta\tilde{t})/k_s} \quad (69)$$

$$\Rightarrow q'_b(2) = hP(T_f - T_p(2\Delta\tilde{t})) \quad (70)$$

$\vdots$

$$T_p(N\Delta\tilde{t}) - T_0 = \frac{S + (T_f - T_0)hPG_{bc}(\Delta\tilde{t})/k_s}{1 + hPG_{bc}(\Delta\tilde{t})/k_s} \quad (71)$$

$$q'_b(N\Delta\tilde{t}) = hP(T_f - T_p(N\Delta\tilde{t})) \quad (72)$$

with

$$S = \sum_{i=1}^{N-1} q'_b(i)(G_{bc}((N - i + 1)\Delta\tilde{t}) - G_{bc}((N - i)\Delta\tilde{t}))/k_s \quad (73)$$

Figs. 8a and 8b show some results for the concentric cylinders subject to mixed boundary conditions with the same

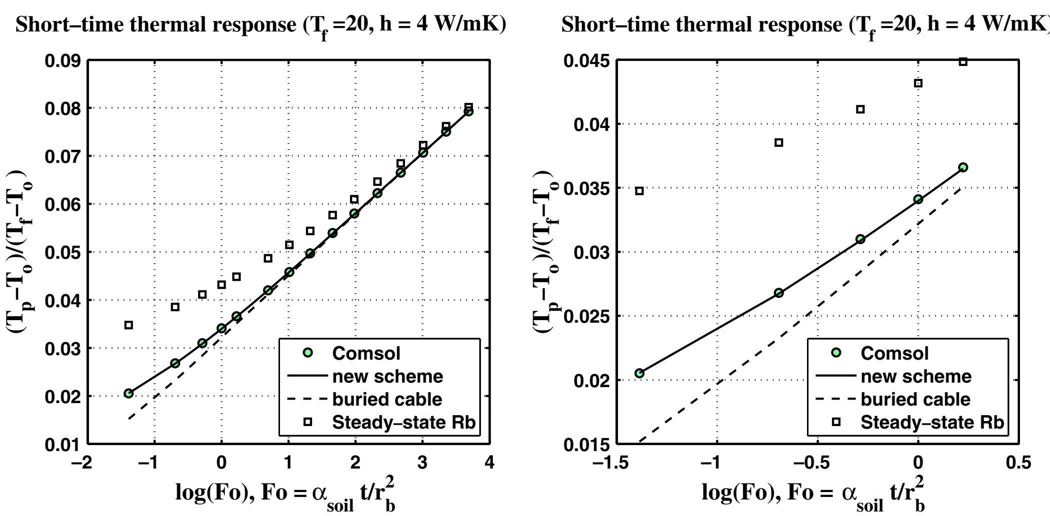

Fig. 8a. Temperature response (concentric cylinder, convection) $\tilde{k} = 1.33$, $\gamma^2 = 1.5$.

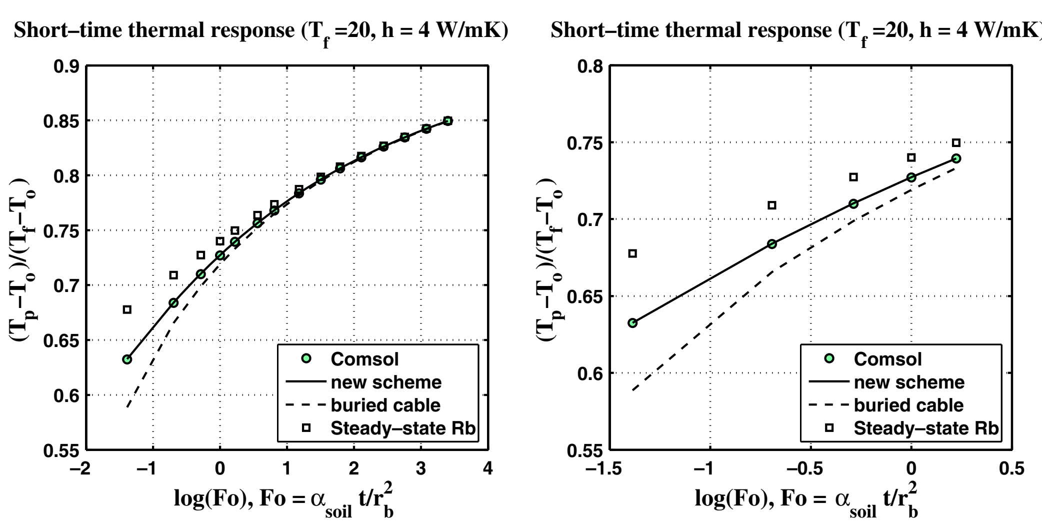

Fig. 8b. Temperature response (concentric cylinder, convection) $\tilde{k} = 0.8$, $\gamma^2 = 0.75$.

value of conductivity and diffusivity as in the fixed heat flux case. As we said in the previous section, the compound solution suggested by Sutton et al. was not considered anymore. As it is seen, (61) is the exact solution of the problem whereas the other approach is a good approximation of the short time thermal response.

4.2.2. Arbitrary fluid temperature

In the case of a variable fluid temperature, the application of the Duhamel's principle can be used. If we assume that the fluid temperature consists of a series of step changes, the solution for this special case is given in Ozisik [13, p. 201]:

$$T_{\mathrm{p}}(\tilde{t}) - T_{\mathrm{f}}(\tilde{t}) = \sum_{j=0}^{N-1} G_{\mathrm{st_c}}(1, \tilde{t} - j\Delta\tilde{t})\Delta(T_0 - T_{f_j}) \tag{74}$$

4.3. U-tube configuration

In the context of a geothermal heat pump, the heat exchanger consists of a U-tube arrangement and in that case, no exact solution is known. However, as it is the case for many models, the one proposed here can be used with the application of an equivalent radius. Some results will be presented here and compared with the other approaches discussed previously, that are the buried cable approximation and the steady-state modeling of the grout. For the expression of the equivalent radius, we chose the value (18), proposed by Sutton et al. [6]. However for the calculation of the borehole resistance, we chose the expression proposed by Hellström [4].

Figs. 9a and 9b show the results for the case where the heat flux is fixed with shank spacing B for two different values of relative conductivity and heat capacity. Two figures are shown for each case where the right one is a zoom for

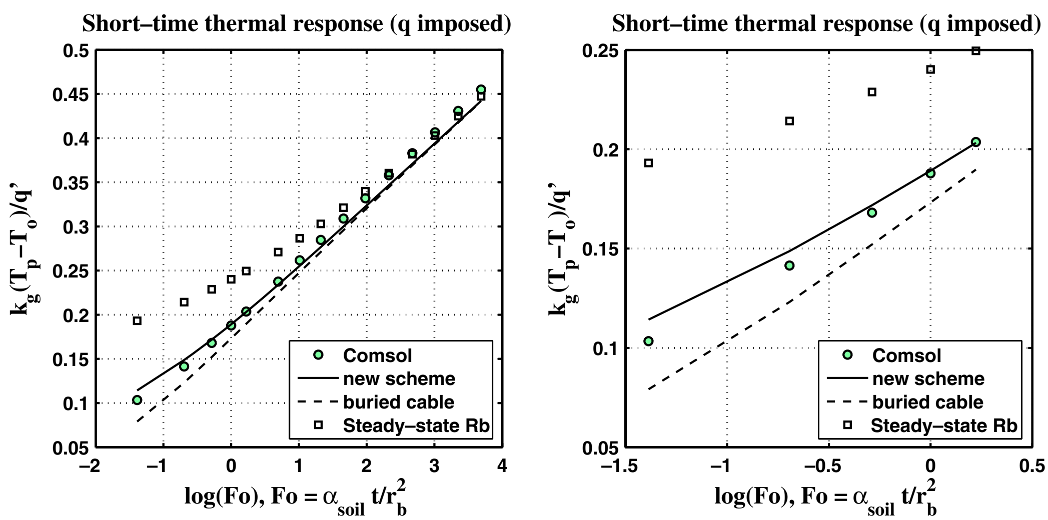

Fig. 9a. Temperature response (U-tube, fixed flux) $\tilde{k} = 1.33$, $\gamma^2 = 1.5$.

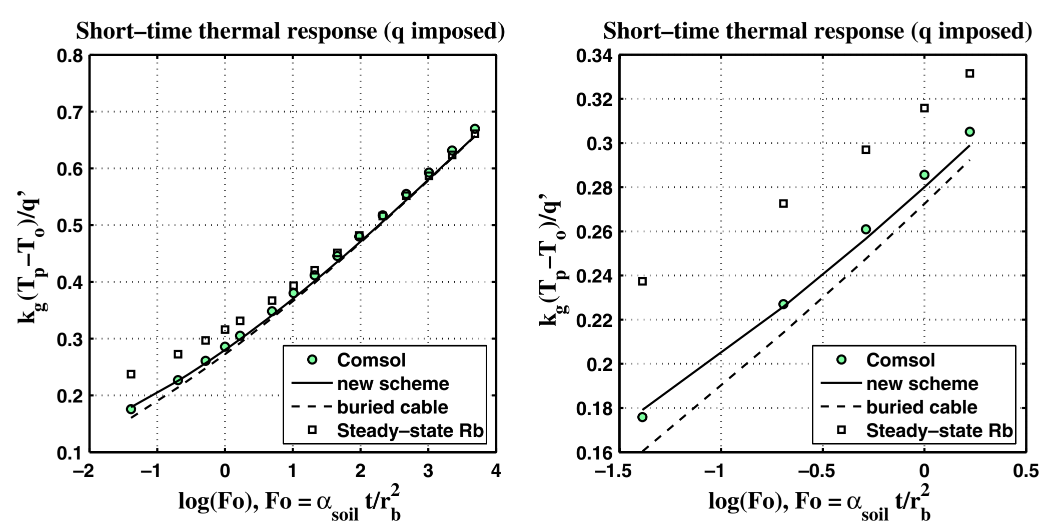

Fig. 9b. Temperature response (U-tube, fixed flux) $\tilde{k} = 0.8$, $\gamma^2 = 0.75$.

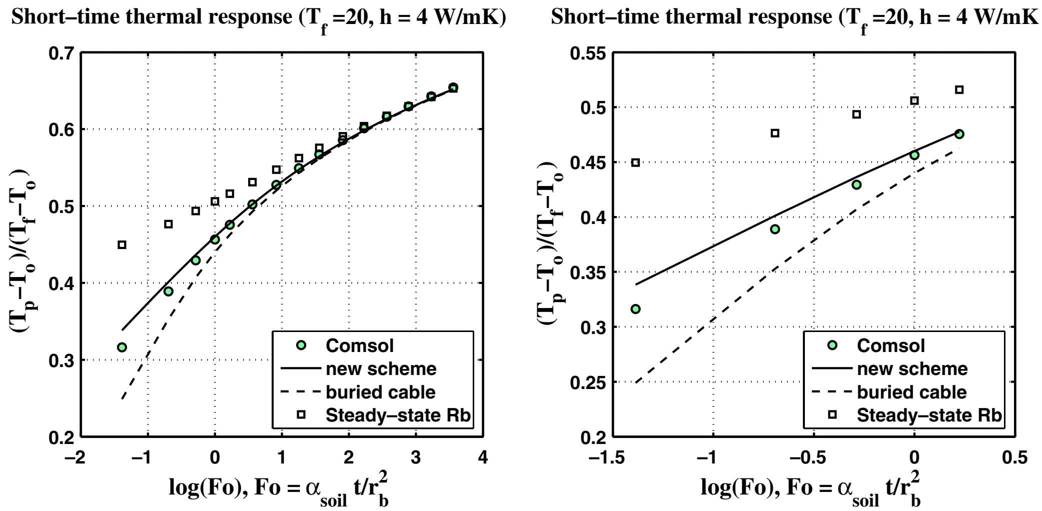

Fig. 10a. Temperature response (U-tube, convection) $\tilde{k} = 1.33$, $\gamma^2 = 1.5$.

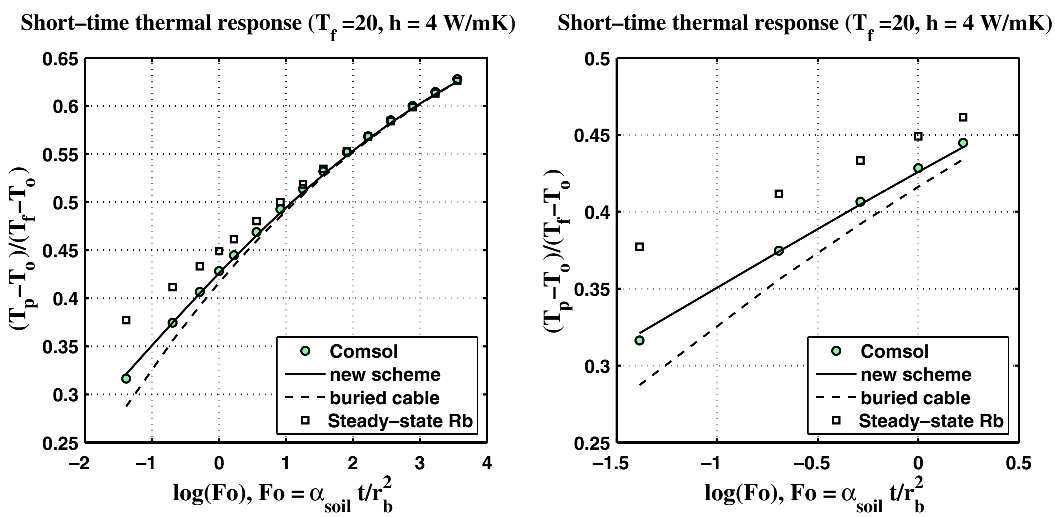

Fig. 10b. Temperature response (U-tube, convection) $\tilde{k} = 0.8$, $\gamma^2 = 0.75$.

very short time. We did the same calculations when mixed boundary conditions are applied. The results are shown in Figs. 10a and 10b for shank spacing B.

5. Conclusion

The solutions presented in this paper are believed to be a good contribution to the modeling of a geothermal heat exchanger for short time transient response. These effects are important when we are interested in the following informations: real time simulations of heat pump systems for time less than an hour, peak load effect for the length calculations of vertical heat exchangers and for the evaluation of the ground thermal properties in short period of time. Although the first aspect can be dealt with by numerical models that take into account the thermal capacities of the borehole, analytical solutions are very attractive for all parts of the design procedures.

It was observed that the model proposed follow very well the numerical results. Although the analytical expressions look rather complicated, these functions were easily implemented and the numerical evaluations of these expressions were very quick.

Appendix A. Zeros of Bessel functions

Inside the integration contour, we have $-\pi < \mathrm{Arg}(s) < \pi$, so $-\pi/2 < \mathrm{Arg}(q) < \pi/2$. In order to prove that (36) has no zeros inside the contour, we have

$$\begin{aligned}\mathrm{den} &= (K_0(q_2 r_b) I_1(q_1 r_b) + \tilde{k}\gamma K_1(q_2 r_b) I_0(q_1 r_b)) K_1(q_1 r_e) \\ &\quad - (K_0(q_2 r_b) K_1(q_1 r_b) - \tilde{k}\gamma K_1(q_2 r_b) K_0(q_1 r_b)) I_1(q_1 r_e) \\ &= (K_1(q_1 r_e) I_1(q_1 r_b) - K_1(q_1 r_b) I_1(q_1 r_e)) K_0(q_2 r_b) \\ &\quad + (I_0(q_1 r_b) K_1(q_2 r_e) + K_0(q_1 r_b) I_1(q_1 r_e)) K_1(q_2 r_b) \tilde{k}\gamma \end{aligned} \quad (75)$$

We can show that these two terms are positive for $|\mathrm{Arg}(q)| < \pi/2$. First of all, it is known [16, p. 377], that $K_n(z)$ has no zeros in that region. Since they are positive on the positive real axis, they remain so in the right hand plane. We need to prove that the two following expressions are positive also:

$$W_1 = K_1(q_1 r_e)I_1(q_1 r_b) - K_1(q_1 r_b)I_1(q_1 r_e) > 0 \tag{76}$$

$$W_2 = I_0(q_1 r_b)K_1(q_2 r_e) + K_0(q_1 r_b)I_1(q_1 r_e) > 0 \tag{77}$$

To do so, we replace the modified Bessel functions by the following expressions:

$$\begin{aligned} I_0(z) &= J_0(z e^{j\pi/2}), \quad I_1(z) = -jJ_1(z e^{j\pi/2}) \\ K_0(z) &= \tfrac{\pi}{2} j H_0(z e^{j\pi/2}), \quad K_1(z) = \tfrac{-\pi}{2} H_1(z e^{j\pi/2}) \end{aligned} \tag{78}$$

So

$$W_1 = \frac{j\pi}{2}(H_1(u_1 r_e)J_1(u_1 r_b) - H_1(u_1 r_b)J_1(u_1 r_e)) \tag{79}$$

$$W_1 = \frac{\pi}{2}(J_1(u_1 r_e)Y_1(u_1 r_b) - J_1(u_1 r_b)Y_1(u_1 r_e)) \tag{80}$$

with $u_1 = q_1 e^{j\pi/2}$, so $0 < \mathrm{Arg}(u_1) < \pi$. Since all the roots of (80) are real and simple [17], this expression has no roots and since it is positive when u_1 is purely imaginary, the real part is always positive. The real part of W_2 is also always positive for $|\mathrm{Arg}(q)| < \pi/2$, since in that region, the real part of I_n and K_n are positive [16] in that region.

To show that (49) has no zeros inside the contour, we proceed to same way:

$$\begin{aligned} (\xi_1\rho_1 - \xi_2\rho_2) &= (K_1(q_1 r_e)I_1(q_1 r_b) - K_1(q_1 r_b)I_1(q_1 r_e))K_0(q_2 r_b)w \\ &+ (K_0(q_1 r_e)I_0(q_1 r_b) - K_0(q_1 r_b)I_0(q_1 r_e))K_1(q_2 r_b)\tilde{k}\gamma \\ &+ (I_0(q_1 r_b)K_1(q_2 r_e) + K_0(q_1 r_b)I_1(q_1 r_e))K_1(q_2 r_b)w\tilde{k}\gamma \\ &+ (I_1(q_1 r_b)K_0(q_2 r_e) + K_1(q_1 r_b)I_0(q_1 r_e))K_0(q_2 r_b) \end{aligned} \tag{81}$$

with $w = \frac{q_1 k_g}{h}$. The first two terms are similar to W_1 for the constant flux case, where as the other two are similar to the W_2 term in the last section.

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